

NORMALITY OF ADJOINTABLE MODULE MAPS

K. SHARIFI

ABSTRACT. Normality of bounded and unbounded adjointable operators are discussed. Suppose T is an adjointable operator between Hilbert C^* -modules which has polar decomposition, then T is normal if and only if there exists a unitary operator \mathcal{U} which commutes with T and T^* such that $T = \mathcal{U}T^*$. Kaplansky's theorem for normality of the product of bounded operators is also reformulated in the framework of Hilbert C^* -modules.

1. INTRODUCTION AND PRELIMINARY.

Normal operators may be regarded as a generalization of a selfadjoint operator T in which T^* need not be exactly T but commutes with T . They form an attractive and important class of operators which play a vital role in operator theory, especially, in spectral theory. In this note we will study bounded and unbounded normal module maps between Hilbert C^* -modules which have polar decomposition. Indeed, for adjointable operator T between Hilbert C^* -modules which has polar decomposition, we demonstrate that T is normal if and only if there exists a unitary operator \mathcal{U} such that $T = \mathcal{U}T^*$. In this situation, $\mathcal{U}T \subseteq T\mathcal{U}$ and $\mathcal{U}T^* \subseteq T^*\mathcal{U}$ (compare [11, Problem 13, page 109]). The results are interesting even in the case of Hilbert spaces.

Suppose T, S are bounded adjointable operators between Hilbert C^* -modules. Suppose T has polar decomposition and T and TS are normal operators. Then we show that ST is a normal operator if and only if S commutes with $|T|$. This fact has been proved by Kaplansky [12] in the case of Hilbert spaces.

Throughout the present paper we assume \mathcal{A} to be an arbitrary C^* -algebra. We deal with bounded and unbounded operators at the same time, so we denote bounded operators by capital letters and unbounded operators by small letters. We use the notations $Dom(\cdot)$, $Ker(\cdot)$ and $Ran(\cdot)$ for domain, kernel and range of operators, respectively.

2000 *Mathematics Subject Classification.* Primary 46L08; Secondary 47A05, 46C05.

Key words and phrases. Hilbert C^* -module, polar decomposition, normal operator, C^* -algebra of compact operators, unbounded operator.

Hilbert C^* -modules are essentially objects like Hilbert spaces, except that the inner product, instead of being complex-valued, takes its values in a C^* -algebra. Although Hilbert C^* -modules behave like Hilbert spaces in some ways, some fundamental Hilbert space properties like Pythagoras' equality, self-duality, and even decomposition into orthogonal complements do not hold. A (right) *pre-Hilbert C^* -module* over a C^* -algebra \mathcal{A} is a right \mathcal{A} -module X equipped with an \mathcal{A} -valued inner product $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathcal{A}$, $(x, y) \mapsto \langle x, y \rangle$, which is \mathcal{A} -linear in the second variable y and has the properties:

$$\langle x, y \rangle = \langle y, x \rangle^*, \quad \langle x, x \rangle \geq 0 \quad \text{with equality only when } x = 0.$$

A pre-Hilbert \mathcal{A} -module X is called a *Hilbert \mathcal{A} -module* if X is a Banach space with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$. A Hilbert \mathcal{A} -submodule W of a Hilbert \mathcal{A} -module X is an orthogonal summand if $W \oplus W^\perp = X$, where W^\perp denotes the orthogonal complement of W in X . We denote by $\mathcal{L}(X)$ the C^* -algebra of all adjointable operators on X , i.e., all \mathcal{A} -linear maps $T : X \rightarrow X$ such that there exists $T^* : X \rightarrow X$ with the property $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x, y \in X$. A bounded adjointable operator $\mathcal{V} \in \mathcal{L}(X)$ is called a *partial isometry* if $\mathcal{V}\mathcal{V}^*\mathcal{V} = \mathcal{V}$, see [16] for some equivalent conditions. For the basic theory of Hilbert C^* -modules we refer to the books [14, 19] and the papers [4, 6].

An unbounded regular operator on a Hilbert C^* -module is an analogue of a closed operator on a Hilbert space. Let us quickly recall the definition. A densely defined closed \mathcal{A} -linear map $t : \text{Dom}(t) \subseteq X \rightarrow X$ is called *regular* if it is adjointable and the operator $1 + t^*t$ has a dense range. Indeed, a densely defined operator t with a densely defined adjoint operator t^* is regular if and only if its graph is orthogonally complemented in $X \oplus X$ (see e.g. [7, 14]). We denote the set of all regular operators on X by $\mathcal{R}(X)$. If t is regular then t^* is regular and $t = t^{**}$, moreover t^*t is regular and selfadjoint. Define $Q_t = (1 + t^*t)^{-1/2}$ and $F_t = tQ_t$, then $\text{Ran}(Q_t) = \text{Dom}(t)$, $0 \leq Q_t = (1 - F_t^*F_t)^{1/2} \leq 1$ in $\mathcal{L}(X)$ and $F_t \in \mathcal{L}(X)$ [14, (10.4)]. The bounded operator F_t is called the bounded transform of regular operator t . According to [14, Theorem 10.4], the map $t \rightarrow F_t$ defines an adjoint-preserving bijection

$$\mathcal{R}(X) \rightarrow \{F \in \mathcal{L}(X) : \|F\| \leq 1 \text{ and } \text{Ran}(1 - F^*F) \text{ is dense in } X\}.$$

Very often there are interesting relationships between regular operators and their bounded transforms. In fact, for a regular operator t , some properties transfer to its bounded transform F_t , and vice versa. Suppose $t \in \mathcal{R}(X)$ is a regular operator, then t is called *normal* iff $\text{Dom}(t) = \text{Dom}(t^*)$ and $\langle tx, tx \rangle = \langle t^*x, t^*x \rangle$ for all $x \in \text{Dom}(t)$. t is called *selfadjoint* iff $t^* = t$ and t is called *positive* iff t is normal and $\langle tx, x \rangle \geq 0$ for all $x \in \text{Dom}(t)$. In particular, a regular operator t is normal (resp., selfadjoint, positive) iff its bounded transform F_t is

normal (resp., selfadjoint, positive). Moreover, both t and F_t have the same range and the same kernel. If $t \in \mathcal{R}(X)$ then $\text{Ker}(t) = \text{Ker}(|t|)$ and $\overline{\text{Ran}(t^*)} = \overline{\text{Ran}(|t|)}$, cf. [13]. If $t \in \mathcal{R}(X)$ is a normal operator then $\text{Ker}(t) = \text{Ker}(t^*)$ and $\overline{\text{Ran}(t)} = \overline{\text{Ran}(t^*)}$.

A bounded adjointable operator T has polar decomposition if and only if $\overline{\text{Ran}(T)}$ and $\overline{\text{Ran}(|T|)}$ are orthogonal direct summands [19, Theorem 15.3.7]. The result has been generalized in Theorem 3.1 of [8] for regular operators. Indeed, for $t \in \mathcal{R}(X)$ the following conditions are equivalent:

- t has a unique polar decomposition $t = \mathcal{V}|t|$, where $\mathcal{V} \in \mathcal{L}(X)$ is a partial isometry for which $\text{Ker}(\mathcal{V}) = \text{Ker}(t)$.
- $X = \text{Ker}(|t|) \oplus \overline{\text{Ran}(|t|)}$ and $X = \text{Ker}(t^*) \oplus \overline{\text{Ran}(t)}$.
- The adjoint operator t^* has polar decomposition $t^* = \mathcal{V}^*|t^*|$.
- The bounded transform F_t has polar decomposition $F_t = \mathcal{V}|F_t|$.

In this situation, $\mathcal{V}^*\mathcal{V}|t| = |t|$, $\mathcal{V}^*t = |t|$ and $\mathcal{V}\mathcal{V}^*t = t$, moreover, we have $\text{Ker}(\mathcal{V}^*) = \text{Ker}(t^*)$, $\text{Ran}(\mathcal{V}) = \overline{\text{Ran}(t)}$ and $\text{Ran}(\mathcal{V}^*) = \overline{\text{Ran}(t^*)}$. That is, $\mathcal{V}\mathcal{V}^*$ and $\mathcal{V}^*\mathcal{V}$ are orthogonal projections onto the submodules $\overline{\text{Ran}(t)}$ and $\overline{\text{Ran}(t^*)}$, respectively.

The above facts and Proposition 1.2 of [7] show that each regular operator with closed range has polar decomposition.

Recall that an arbitrary C^* -algebra of compact operators \mathcal{A} is a c_0 -direct sum of elementary C^* -algebras $\mathcal{K}(H_i)$ of all compact operators acting on Hilbert spaces H_i , $i \in I$, cf. [2, Theorem 1.4.5]. Generic properties of Hilbert C^* -modules over C^* -algebras of compact operators have been studied systematically in [1, 3, 7, 8, 10] and references therein. If \mathcal{A} is a C^* -algebra of compact operators then for every Hilbert \mathcal{A} -module X , every densely defined closed operator $t : \text{Dom}(t) \subseteq X \rightarrow X$ is automatically regular and has polar decomposition, cf. [7, 8, 10].

The stated results also hold for bounded adjointable operators, since $\mathcal{L}(X)$ is a subset of $\mathcal{R}(X)$. The space $\mathcal{R}(X)$ from a topological point of view are studied in [15, 17, 18].

2. NORMALITY

Proposition 2.1. *Suppose $T \in \mathcal{L}(X)$ admits the polar decomposition $T = \mathcal{V}|T|$ and $S \in \mathcal{L}(X)$ is an arbitrary operator which commutes with T and T^* . Then \mathcal{V} and $|T|$ commute with S and S^* .*

Proof. It follows from the hypothesis that $(T^*T)S = S(T^*T)$ which implies $|T|S = S|T|$, or equivalently $|T|S^* = S^*|T|$. Using the commutativity of S with T and $|T|$, we get

$$(S\mathcal{V} - \mathcal{V}S)|T| = S\mathcal{V}|T| - \mathcal{V}|T|S = ST - TS = 0.$$

That is, $S\mathcal{V} - \mathcal{V}S$ acts as zero operator on $\overline{\text{Ran}(|T|)}$. If $x \in \text{Ker}(|T|) = \text{Ker}(\mathcal{V})$ then $|T|x = \mathcal{V}x = 0$, consequently $|T|Sx = S|T|x = 0$. Then $Sx \in \text{Ker}(|T|) = \text{Ker}(\mathcal{V})$, therefore, $S\mathcal{V} - \mathcal{V}S$ acts as zero operator on $\text{Ker}(|T|)$ too. We obtain

$$S\mathcal{V} - \mathcal{V}S = 0 \quad \text{on} \quad X = \text{Ker}(|T|) \oplus \overline{\text{Ran}(|T|)}.$$

The statement $S^*\mathcal{V} - \mathcal{V}S^* = 0$ on $X = \text{Ker}(|T|) \oplus \overline{\text{Ran}(|T|)}$ can be deduced from the commutativity of S with T^* and $|T|$ in the same way. \square

Corollary 2.2. *Suppose $T \in \mathcal{L}(X)$ is a normal operator which admits the polar decomposition $T = \mathcal{V}|T|$ then \mathcal{V} and $|T|$ commute with the operators T, T^*, \mathcal{V} and \mathcal{V}^* . In particular, \mathcal{V} is a unitary operator on $\overline{\text{Ran}(T)} = \overline{\text{Ran}(T^*)}$.*

The results follow from Proposition 2.1, Proposition 3.7 of [14] and the fact that $\mathcal{V}\mathcal{V}^*T = \mathcal{V}^*\mathcal{V}T = T$.

Corollary 2.3. *Suppose $T \in \mathcal{L}(X)$ admits the polar decomposition $T = \mathcal{V}|T|$. Then T is a normal operator if and only if there exists a unitary operator $\mathcal{U} \in \mathcal{L}(X)$ commuting with $|T|$ such that $T = \mathcal{U}|T|$. In this situation, \mathcal{U} also commutes with T and T^* .*

Proof. Suppose T is a normal operator then $\text{Ker}(T) = \text{Ker}(T^*)$ and $\overline{\text{Ran}(T)} = \overline{\text{Ran}(T^*)}$. For every $x \in X = \text{Ker}(T) \oplus \overline{\text{Ran}(T^*)}$ we define

$$\mathcal{U}x = \begin{cases} x & \text{if } x \in \text{Ker}(T) \\ \mathcal{V}x & \text{if } x \in \overline{\text{Ran}(T^*)}, \end{cases}$$

$$\mathcal{W}x = \begin{cases} x & \text{if } x \in \text{Ker}(T^*) \\ \mathcal{V}^*x & \text{if } x \in \overline{\text{Ran}(T)}. \end{cases}$$

Then $\langle \mathcal{U}x, y \rangle = \langle x, \mathcal{W}y \rangle$ for all $x, y \in X$, that is, $\mathcal{W} = \mathcal{U}^*$. For each $x = x_1 + x_2 \in X$ with $x_1 \in \text{Ker}(T)$ and $x_2 \in \overline{\text{Ran}(T^*)}$ we have

$$\mathcal{U}\mathcal{U}^*x = \mathcal{U}(x_1 + \mathcal{V}^*x_2) = x_1 + \mathcal{V}\mathcal{V}^*x_2 = x.$$

Hence, $\mathcal{U}\mathcal{U}^* = 1$ on X . We also have $\mathcal{U}^*\mathcal{U} = 1$ and $T = \mathcal{U}|T|$ on X . Commutativity of \mathcal{U} with T, T^* and $|T|$ follows from the commutativity of \mathcal{V} with T, T^* and $|T|$.

Conversely, suppose $T = \mathcal{U}|T|$ for a unitary operator $\mathcal{U} \in \mathcal{L}(X)$ which commutes with $|T|$. Then $T^* = |T|\mathcal{U}^*$ and so $TT^* = \mathcal{U}|T||T|\mathcal{U}^* = |T|\mathcal{U}|T|\mathcal{U}^* = T^*T$. \square

Corollary 2.4. *Suppose $T \in \mathcal{L}(X)$ admits the polar decomposition $T = \mathcal{V}|T|$. Then T is a normal operator if and only if there exists a unitary operator $\mathcal{U} \in \mathcal{L}(X)$ such that $T = \mathcal{U}T^*$. In this situation, \mathcal{U} commutes with T and T^* .*

Proof. Suppose T is a normal operator then $|T| = |T^*| = \mathcal{V}T^*$ and so $T = \mathcal{V}|T| = \mathcal{V}|T^*| = \mathcal{V}^2T^*$. For $x \in X$ we define

$$\mathcal{U}x = \begin{cases} x & \text{if } x \in \text{Ker}(T) \\ \mathcal{V}^2x & \text{if } x \in \overline{\text{Ran}(T^*)}. \end{cases}$$

Then, as in the proof of Corollary 2.3,

$$\mathcal{U}^*x = \begin{cases} x & \text{if } x \in \text{Ker}(T^*) \\ \mathcal{V}^{*2}x & \text{if } x \in \overline{\text{Ran}(T)}, \end{cases}$$

which implies \mathcal{U} is unitary and $T = \mathcal{U}T^*$. Commutativity of \mathcal{U} with T and T^* follows from the commutativity of \mathcal{V} with T and T^* .

Conversely, suppose $T = \mathcal{U}T^*$ for a unitary operator $\mathcal{U} \in \mathcal{L}(X)$. Then $T^* = (\mathcal{U}T^*)^* = T\mathcal{U}^*$ and so $T^*T = T\mathcal{U}^*\mathcal{U}T^* = TT^*$. \square

If the normal operator $T \in \mathcal{L}(X)$ has closed range, one can find a shorter proof for the above result.

Theorem 2.5. *Suppose $t \in \mathcal{R}(X)$ admits the polar decomposition $t = \mathcal{V}|t|$. Then t is a normal operator if and only if there exists a unitary operator $\mathcal{U} \in \mathcal{L}(X)$ such that $t = \mathcal{U}t^*$. In this situation, $t\mathcal{U} = \mathcal{U}t$ and $t^*\mathcal{U} = \mathcal{U}t^*$ on $\text{Dom}(t) = \text{Dom}(t^*)$.*

Proof. Recall that t admits the polar decomposition $t = \mathcal{V}|t|$ if and only if its bounded transform F_t admits the polar decomposition $F_t = \mathcal{V}|F_t|$, furthermore, t is a normal operator if and only if its bounded transform F_t is a normal operator.

Suppose t is a normal operator then there exists a unitary operator $\mathcal{U} \in \mathcal{L}(X)$ such that $tQ_t = F_t = \mathcal{U}F_t^* = \mathcal{U}F_t^* = \mathcal{U}t^*Q_t^* = \mathcal{U}t^*Q_t$. Since $Q_t : X \rightarrow \text{Ran}(Q_t) = \text{Dom}(t)$ is invertible, we obtain $t = \mathcal{U}t^*$.

Conversely, suppose $t = \mathcal{U}t^*$ for a unitary operator $\mathcal{U} \in \mathcal{L}(X)$. Then, in view of Remark 2.1 of [8], we have $t^* = (\mathcal{U}t^*)^* = t^{**}\mathcal{U}^* = t\mathcal{U}^*$ on $\text{Dom}(t^*)$ and so $t^*t = t\mathcal{U}^*\mathcal{U}t^* = tt^*$.

According to Corollary 2.4 and the first paragraph of the proof, the unitary operator \mathcal{U} commutes with F_t and F_t^* . Thus for every polynomial p we have $\mathcal{U}p(F_t^*F_t) = p(F_t^*F_t)\mathcal{U}$ and so for every continuous function $p \in \mathbf{C}[0, 1]$ we have $\mathcal{U}p(F_t^*F_t) = p(F_t^*F_t)\mathcal{U}$. In particular, $\mathcal{U}(1 - F_t^*F_t)^{1/2} = (1 - F_t^*F_t)^{1/2}\mathcal{U}$ which implies $\mathcal{U}Q_t = Q_t\mathcal{U}$. This fact together with

the equality $F_t \mathcal{U} = \mathcal{U} F_t$ imply that $t \mathcal{U} Q_t = t Q_t \mathcal{U} = \mathcal{U} t Q_t$. Again by invertibility of the map $Q_t : X \rightarrow \text{Ran}(Q_t) = \text{Dom}(t)$ we obtain $t \mathcal{U} = \mathcal{U} t$ on $\text{Dom}(t)$. To demonstrate the second equality we have $\mathcal{U}^* t = \mathcal{U}^* \mathcal{U} t^* = t^*$ which yields $t^* \mathcal{U} = (\mathcal{U}^* t)^* = t^{**} = t = \mathcal{U} t^*$ on $\text{Dom}(t^*)$. \square

Corollary 2.6. *Suppose $t \in \mathcal{R}(X)$ has closed range. Then t is a normal operator if and only if there exists a unitary operator $\mathcal{U} \in \mathcal{L}(X)$ such that $t = \mathcal{U} t^*$. In this situation, $t \mathcal{U} = \mathcal{U} t$ and $t^* \mathcal{U} = \mathcal{U} t^*$ on $\text{Dom}(t) = \text{Dom}(t^*)$.*

The proof immediately follows from Theorem 2.5, Proposition 1.2 of [7] and Theorem 3.1 of [8].

Corollary 2.7. *Suppose X is a Hilbert space (or a Hilbert C^* -module over an arbitrary C^* -algebra of compact operators) and $t : \text{Dom}(t) \subseteq X \rightarrow X$ is a densely defined closed operator. Then t is a normal operator if and only if there exists a unitary operator $\mathcal{U} \in \mathcal{L}(X)$ such that $t = \mathcal{U} t^*$. In this situation, $t \mathcal{U} = \mathcal{U} t$ on $\text{Dom}(t) = \text{Dom}(t^*)$.*

Consider two normal operators T and S on a Hilbert space it is known that, in general, TS is not normal. Historical notes and several versions of the problem are investigated in [9]. Kaplansky has shown that it may be possible that TS is normal while ST is not. Indeed, he has shown that if T and TS are normal, then ST is normal if and only if S commutes with $|T|$, cf. [12]. We generalize his result for bounded adjointable operator on Hilbert C^* -modules. For this aim we need the Fuglede-Putnam theorem for bounded adjointable operators on Hilbert C^* -modules. Using Theorem 4.1.4.1 of [5] for the unital C^* -algebra $\mathcal{L}(X)$, we obtain:

Theorem 2.8. (Fuglede-Putnam) *Assume that T, S and A are bounded adjointable operator in $\mathcal{L}(X)$. Suppose T and S are normal and $TA = AS$, then $T^*A = AS^*$.*

Corollary 2.9. *Let $T, S \in \mathcal{L}(X)$ be such that T and TS are normal and T has polar decomposition. ST is normal if and only if S commutes with $|T|$.*

Proof. Suppose ST and T are normal operators and $A = TS$ and $B = ST$, then $AT = TB$. In view of the Theorem 2.8, $A^*T = TB^*$, that is, $S^*T^*T = TT^*S^*$, and taking into account the normality of T , we find S^* commutes with T^*T . Therefore, $S^*|T| = |T|S^*$ and so S commutes with $|T|$ by the Fuglede-Putnam theorem.

Conversely, suppose S commutes with $|T|$. Then the normal operator T has a representation $T = \mathcal{U} |T|$ in which $\mathcal{U} \in \mathcal{L}(X)$ is unitary and commutes with $|T|$. Therefore,

$$\mathcal{U}^* TS \mathcal{U} = \mathcal{U}^* \mathcal{U} |T| S \mathcal{U} = S |T| \mathcal{U} = S \mathcal{U} |T| = ST.$$

The operator ST is normal as an operator which is unitary equivalent with the normal operator TS . \square

REFERENCES

- [1] Lj. Arambašić, Another characterization of Hilbert C^* -modules over compact operators, *J. Math. Anal. Appl.* **344** (2008), no. 2, 735-740.
- [2] W. Arveson, *An Invitation to C^* -algebras*, Springer, New York, 1976.
- [3] D. Bakić and B. Guljaš, Hilbert C^* -modules over C^* -algebras of compact operators, *Acta Sci. Math. (Szeged)* **68** (2002), no. 1-2, 249-269.
- [4] D. Bakić and B. Guljaš, On a class of module maps of Hilbert C^* -modules, *Math. Commun.* **7** (2002), no. 2, 177-192.
- [5] C. Constantinescu, *C^* -algebras. Vol. 3: General theory of C^* -algebras*, North-Holland, Amsterdam, 2001.
- [6] M. Frank, Geometrical aspects of Hilbert C^* -modules, *Positivity* **3** (1999), 215-243.
- [7] M. Frank and K. Sharifi, Adjointability of densely defined closed operators and the Magajna-Schweizer theorem, *J. Operator Theory* **63** (2010), 271-282.
- [8] M. Frank and K. Sharifi, Generalized inverses and polar decomposition of unbounded regular operators on Hilbert C^* -modules, *J. Operator Theory* **64** (2010), 377-386.
- [9] A. Gheondea, When are the products of normal operators normal?, *Bull. Math. Soc. Roum. Math.*, **52** (2009), 129-150.
- [10] B. Guljaš, Unbounded operators on Hilbert C^* -modules over C^* -algebras of compact operators, *J. Operator Theory* **59** (2008), no. 1, 179-192.
- [11] R. A. Horn and C. Johnson, *Matrix Analysis*, Cambridge Univ. Press, 1985.
- [12] I. Kaplansky, Products of normal operators, *Duke Math. J.* **20** (1953), 257-260.
- [13] J. Kustermans, The functional calculus of regular operators on Hilbert C^* -modules revisited, available at arXiv:funct-an/9706007 v1 20 Jun 1997.
- [14] E. C. Lance, *Hilbert C^* -Modules*, LMS Lecture Note Series 210, Cambridge Univ. Press, 1995.
- [15] K. Sharifi, The gap between unbounded Regular operators, to appear in *J. Operator Theory*, available on arXiv:math.OA/0901.1891 v1 13 Jan 2009.
- [16] K. Sharifi, Descriptions of partial isometries on Hilbert C^* -modules, *Linear Algebra Appl.* **431** (2009), 883-887.
- [17] K. Sharifi, Topological approach to unbounded operators on Hilbert C^* -modules, to appear in *Rocky Mountain J. Math.*
- [18] K. Sharifi, Continuity of the polar decomposition for unbounded operators on Hilbert C^* -modules, *Glas. Mat. Ser. III.* **45** (2010), 505-512
- [19] N. E. Wegge-Olsen, *K -theory and C^* -algebras: a Friendly Approach*, Oxford University Press, Oxford, England, 1993.

KAMRAN SHARIFI,
DEPARTMENT OF MATHEMATICS, SHAHROOD UNIVERSITY OF TECHNOLOGY, P. O. Box 3619995161-
316, SHAHROOD, IRAN

E-mail address: `sharifi.kamran@gmail.com` and `sharifi@shahroodut.ac.ir`